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# Quantum Mechanics of 'Free' Spin-1/2 Particles in an Expanding Universe

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*Abstract*

This paper deals with the gravi-quantum mechanical interaction on the level of the first quantisation and in the framework of a metric theory of gravitation (no field quantisation). The interaction is introduced by embedding the quantum mechanics of the otherwise unaffected (i.e. 'free') spin-$\frac{1}{2}$ particle in the given curved space-time of the 3-flat expanding Robertson–Walker universe. The metric acts thereby as an external field. The corresponding Hilbert space formalism is established in interpreting the generally covariant theory of the Dirac field in the Riemann space in question as the Dirac representation of the spin-$\frac{1}{2}$ particle in the Schrödinger picture. The evolution operator is then extracted out of the general relativistic Dirac equation, while contractions of the symmetric energy momentum tensor with the tetrad vectors of the reference system lead to the operators of energy, linear momentum and total angular momentum. The temporal behaviour of the corresponding expectation values is calculated.

## 1. *Method and Basic Assumptions*

This paper deals with the *gravi-quantum mechanical interaction* on the level of first quantisation and in the framework of a metric theory of gravitation. The interaction is thereby introduced by embedding the quantum mechanics of the particle in a given curved space-time, which represents the unquantised 'external' (i.e. unaffected) 'gravitational' field. The main purpose of this paper is to establish first of all the corresponding *quantum mechanical formalism* (i.e. Hilbert space, dynamical equation, fundamental operators) for the example of otherwise unaffected (i.e. 'free') *Dirac particles in a 3-flat expanding universe.* Later, as an application of the formalism, the tempiral behaviour of the mean values of energy, linear momentum and total angular momentum are calculated. Further applications will be given in a subsequent paper (Audretsch, 1973).

This treatment is of interest for the following reasons: It is usual to consider General Relativity on the one hand and Quantum Mechanics on the other as describing entirely different parts of physical reality that there is neither the necessity nor the possibility of bringing these two very conceptually different theories together. Leaving aside the fact that this situation is unsatisfactory from the point of view of the unity of physics (gravitation as space-time geometry is universally influencing every process), it is furthermore dangerous to use results of 'everyday' physics in circumstances where strong gravitational fields and high accelerations are characteristic. The physical behaviour of matter under such extreme conditions is largely unknown to the present day. Strong space-time curvature may cause physically relevant deviations from special relativistic quantum mechanics in the exterior and interior of massive stars, by the influence of gravitational waves and at the very early rapidly expanding stages of the universe. Simple objects for which the influence of the expansion of the universe on matter can be studied quantum mechanically are the freely moving Dirac particles.

*Restrictions and Assumptions*

We will make the following restrictions and assumptions:

($A_1$) Quantum theory: Level of the first quantisation (i.e. no field quantisation).

($A_2$) Metric theory of gravitation: An unquantised metric acts as an external field. The gravitational field caused by the particle itself is therefore neglected ('test'-particle).

($A_3$) External fields: Apart from the metric field there are no further fields acting on the particle.

($A_4$) Universe: The metric field of the curved space-time is that of a homogeneous and isotropic universe as described by the 3-flat Robertson-Walker line-element.

($A_5$) Topology: Leaving apart the singularity at the origin of time, the manifold has Euclidian topology $E^{(4)}$.

It is difficult to avoid the 'test'-particle approach of $A_2$ because the probability interpretation of quantum mechanics prevents a consistent way of coupling a classical metric field to quantised matter (compare for the quantised Dirac field, Arnowitt & Kannenberg (1967)). On the other hand, the attempts to quantise the gravitational field have not yet reached a satisfactory stage (compare, e.g., Brill & Gowdy (1970)). By $A_4$ and $A_5$ the analogy to the Minkowski space treatment is made easy. The effects of a non-Minkowskian topology, e.g. the discretisation of eigenvalue spectra, are avoided and the consequences of the expansion are stressed. This may be justified by the fact that in reality an electron wave function ends at 'the wall of the laboratory' while the time dependence of the metric remains locally effective.

*Method.* The formulation of the gravi-quantum mechanical interaction given below is largely determined by the characteristic properties of the theories $A_1$ and $A_2$ which are to be combined. On the one hand we have as starting point the metric field of the theory $A_2$ which represents the unification of the two basic concepts: *gravitation* and *space-time geometry.* Accordingly we have to incorporate metric effects at two points:

(i) The metric influences, as a guiding field, the motion of particles and physical fields.

(ii) A measurement can at least in principle be reduced to a measurement of space and time intervals between events. These distances, and therefore the measuring results, are determined by the metric.

On the other hand we have to incorporate as characteristics of the quantum theory $A_1$ its *probability interpretation* and *dynamics*, i.e. the fact that the change in time of a quantum mechanical system takes place in a threefold way:

(a) The physical state changes continuously under the influence of external fields. This is described by an evolution operator.
(b) The operators representing observables can be explicitly time-dependent.
(c) Measurement of an observable changes the state into an eigenstate of the corresponding operator. The measuring device is thereby as a macroscopic object part of classical physics.

We fulfill the requirements stated above *in postulating that the covariant theory of the Dirac field* $\Psi(x)$ *in the space-time in question represents the Dirac spin-position representation of the spin-*$\frac{1}{2}$ *particle in the Schrödinger picture.* The Dirac position operator refers in contrast to the Newman-Wigner position operator of the Foldy-Wouthuysen representation to a particular space-time point ('point' position operator). The corresponding $\delta$-function eigenstates contain necessarily positive and negative energy components.‡ This additional degree of freedom makes a four-dimensional spin space necessary. The Newman-Wigner and other position operators have the more physical properties but describe a frame dependent *average* position of the extansion $\hbar/mc$ ('mean' position operator). Because the influence of the metric is a strictly local one, we have to base the incorporation of the gravitational interaction on the Dirac position operator. Therefore, because an external field is present, we cannot exclude negative energy states from the very first. The particle concept has to be used in this wider sense.

In order to be able to discuss the behaviour of the energy states of different sign, to pass to another picture or to change the representation by means of, for example, a Foldy-Wouthuysen type transformation, we have to formulate the interaction dynamics and the corresponding operators of energy and momentum in the framework of a Hilbert space formalism. In Section 4 we construct the abstract Hilbert space H of the states $|\Psi\rangle$ of the system and establish the connection

$$\Psi(x) \leftrightarrow |\Psi\rangle \tag{1.1}$$

In Section 5 we unite (i) and (a) in extracting the evolution operator out of the general relativistic Dirac equation

$$\text{covariant Dirac equation} \leftrightarrow \text{evolution operator} \tag{1.2}$$

Finally, to combine the points (ii), (b) and the classical description of the measuring device (compare (c)), we introduce the observer field $h^{\alpha}_{(4)}$ and a

‡ Newton & Wigner (1949), Foldy & Wouthuysen (1950), Feshbach & Villars (1958), Fleming (1965).

family of hypersurfaces $\Omega$ in the space-time and postulate as a correspondence principle the equality of the mean value $\langle\Psi|\mathscr{A}|\Psi\rangle$ of an operator $\mathscr{A}$ and the corresponding classical expression for the observable, as constructed with the general relativistic symmetric energy-momentum tensor:

$$\int_{\Omega} T_{\alpha\beta} a^{\alpha} h^{\beta}_{(4)}\, d^3 V = \langle\Psi|\mathscr{A}|\Psi\rangle \tag{1.3}$$

($a^{\alpha}$ to be chosen appropriate). The fundamental operators are extracted in Sections 6–8. As starting points we summarise in Sections 2 and 3 parts of the tetrad approach to 4-spinor calculus in Riemann space and some properties of the cosmological space-time, as far as this is necessary for further reference. Section 2 will show that the incorporation of space-time curvature in quantum mechanics via the Dirac representation is based on the *principle of minimal coupling.* A discussion of alternative approaches will be given in the Appendix.

## 2. *Dirac Fields in Curved Space-Time*

Pseudo-orthonormal *tetrad fields*‡

$$h^{\alpha}_{a} h^{\beta}_{b} g_{\alpha\beta} = \eta_{ab} = \operatorname{diag}(-1, -1, -1, +1),\; h^{\alpha}_{a} h^{\beta}_{b} \eta^{ab} = g^{\alpha\beta} \tag{2.1a, b}$$

with $h^{\alpha}_{(4)}$ adjusted as tangent vector to a time-like congruence, form an important mathematical tool for the introduction of generally covariant spinor fields, the formulation of a general relativistic theory of measurement (3 + 1 formalism) and the description of cosmological models (compare e.g., Treder (1971)).

The tetrads $h^{\alpha}_{a}(x)$ span a local tangent space at every point. The components of a 4-spinor field $\boldsymbol{\Psi}(x)$ in a Riemann space behave like scalars under coordinate transformations. $\boldsymbol{\Psi}(x)$ is defined locally with respect to the tangent space. A proper orthochronous Lorentz transformation in this tangent space, as represented by a tetrad rotation, induces a homomorphic, unimodular, *position dependent spin-transformation* according to §

$$h^{\alpha'}_{a} = \Omega_{a}{}^{b} h^{\alpha}_{b}, \qquad \boldsymbol{\Psi}' = \mathbf{S}\boldsymbol{\Psi} \qquad \text{with} \qquad \Omega_{a}{}^{b}\boldsymbol{\gamma}^{a} = \mathbf{S}\boldsymbol{\gamma}^{b}\mathbf{S}^{-1} \tag{2.2}$$

‡ *Notations and conventions:* A definition is indicated by : =. Signature of the metric tensor $g^{\alpha\beta}$: $(- - - +)$. $\alpha, \beta, \ldots = 1, \ldots, 4$ and $\hat{\alpha}, \hat{\beta}, \ldots = 1, 2, 3$ are tensor indices raised and lowered with $g^{\alpha\beta}$. $a, b, \ldots = 1, \ldots, 4$ and $\hat{a}, \hat{b}, \ldots = 1, 2, 3$ are tetrad indices raised and lowered with $\eta^{ab} = \operatorname{diag}(-1, -1, -1, +1)$. The corresponding geometrical object is a Riemannian scalar with regard to $a, b, \ldots$. $y := \{y^{\hat{a}}\}$, $M^{(1)} := M^{a=1}$, $M^{1} := M^{\alpha=1}$. $A, B, \ldots = 1, \ldots, 4$ are spinor indices. $\varphi_A, \gamma^{a}_{AB}, \ldots$ are the components of $\boldsymbol{\varphi}, \boldsymbol{\gamma}^{a}, \ldots$. The latter are connected by matrix multiplication. Covariant derivatives of tensors and spinors are denoted by $_{\|\alpha}$.

$$M_{(\alpha\beta)} := \tfrac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha}), \qquad M_{[\alpha\beta]} := \tfrac{1}{2}(M_{\alpha\beta} - M_{\beta\alpha}).$$

§ For a survey of the theory of spinors in curved space-time and a review of the literature see e.g. Bade & Jehle (1953), Brill & Wheeler (1957), Cap, Majerotto & Unteregger (1966) and Schmutzer (1968).

($\Omega_a{}^b = \Omega_a{}^b(x)$, $\mathbf{S} = \mathbf{S}(x)$) where $\boldsymbol{\gamma}^a$ are the usual Dirac matrices

$$\boldsymbol{\gamma}^{(a}\boldsymbol{\gamma}^{b)} = \eta^{ab}, \qquad \frac{\partial \boldsymbol{\gamma}^a}{\partial x^\alpha} = 0. \tag{2.3a, b}$$

The connection between the infinitesimal transformations is given explicitly by

$$\Omega_a{}^b = \delta_a^b + \epsilon_a^b, \qquad \epsilon_{(ab)} = 0, \qquad \mathbf{S} = \mathbf{1} + \mathbf{R}, \qquad \mathbf{R} = \tfrac{1}{4}\epsilon_{ab}\boldsymbol{\gamma}^a\boldsymbol{\gamma}^b \tag{2.4}$$

The position dependent transformation behaviour of $\boldsymbol{\Psi}(x)$ makes it necessary to define a non-trivial parallel propagation of $\boldsymbol{\Psi}(x)$ (compare Weyl (1929)) and accordingly to introduce a *covariant spinor derivative*

$$\boldsymbol{\Psi}_{\|\alpha} = \frac{\partial \boldsymbol{\Psi}}{\partial x^\alpha} + \boldsymbol{\Gamma}_\alpha \boldsymbol{\Psi} \tag{2.5}$$

The spinor affinity $\boldsymbol{\Gamma}_\alpha$ can thereby be formulated in terms of the respective tetrad field

$$\boldsymbol{\Gamma}_\alpha = \tfrac{1}{4} h^\kappa_{a\|\alpha} h_{\kappa b} \boldsymbol{\gamma}^b \boldsymbol{\gamma}^a \tag{2.6}$$

The theory of the *generally covariant Dirac field* can be derived from a Lagrange density (no non-metric external fields according to $A_3$)

$$\mathscr{L} = \sqrt{(-g)}\,\frac{\hbar i}{2}\left(\bar{\boldsymbol{\Psi}}_{\|\mu}\boldsymbol{\gamma}^\mu\boldsymbol{\Psi} - \bar{\boldsymbol{\Psi}}\boldsymbol{\gamma}^\mu\boldsymbol{\Psi}_{\|\mu} - i\,\frac{2mc}{\hbar}\,\bar{\boldsymbol{\Psi}}\boldsymbol{\Psi}\right) \tag{2.7}$$

which is obtained from the special relativistic one by replacing partial derivatives by covariant derivatives (*minimal coupling*)‡. The generalised Dirac metrices are introduced by

$$\boldsymbol{\gamma}^\alpha := h_a^\alpha \boldsymbol{\gamma}^a, \qquad \boldsymbol{\gamma}^{(\alpha}\boldsymbol{\gamma}^{\beta)} = g^{\alpha\beta} \tag{2.8a, b}$$

Their covariant derivative vanishes because of (2.6)

$$\boldsymbol{\gamma}^\alpha{}_{\|\epsilon} := \frac{\partial \boldsymbol{\gamma}^\alpha}{\partial x^\epsilon} + \boldsymbol{\Gamma}^\alpha_{\epsilon\kappa}\boldsymbol{\gamma}^\kappa + \boldsymbol{\Gamma}_\epsilon\boldsymbol{\gamma}^\alpha - \boldsymbol{\gamma}^\alpha\boldsymbol{\Gamma}_\epsilon = 0 \tag{2.9}$$

The adjoined spinor $\bar{\boldsymbol{\Psi}}$ is defined with the aid of Hermitesing matrix $\mathbf{B}$

$$\bar{\boldsymbol{\Psi}} := \boldsymbol{\Psi}^\dagger \mathbf{B}, \qquad \bar{\boldsymbol{\Psi}}' = \bar{\boldsymbol{\Psi}}\,\mathbf{S}^{-1}, \qquad \bar{\boldsymbol{\Psi}}_{\|\alpha} = \frac{\partial \bar{\boldsymbol{\Psi}}}{\partial x^\alpha} - \bar{\boldsymbol{\Psi}}\boldsymbol{\Gamma}_\alpha \tag{2.10a–c}$$

‡ This has the character of an *equivalence principle* generalised to the spinorial probability field $\boldsymbol{\Psi}$: With regard to a *local Lorentz system* at an arbitrary point $P$ as defined by a reference frame at $P$ with $h_a^\alpha(P)_{\|\epsilon} = 0$, the differential equations at $P$ for $\Psi$ take the special relativistic form. Tetrad rotations leading to the general case $h_a^\alpha(P)_{\|\epsilon} \neq 0$ are connected with position dependent spin transformations which result in the replacement of the partial by the covariant spinor derivative according to the definition of the latter.

which for special $\Upsilon^a$, with

$$(\Upsilon^{(4)}\Upsilon^a)^\dagger = \Upsilon^{(4)}\Upsilon^a, \qquad \Upsilon^{(4)\dagger} = \Upsilon^{(4)} \tag{2.11}$$

can chosen to be‡

$$\mathbf{B} = \Upsilon^{(4)} \tag{2.12}$$

From (2.7) we get the Dirac equations as field equations

$$\frac{\delta\mathscr{L}}{\delta\boldsymbol{\Psi}} = 0, \qquad \frac{\delta\mathscr{L}}{\delta\overline{\boldsymbol{\Psi}}} = 0, \qquad i\Upsilon^\mu\boldsymbol{\Psi}_{\|\mu} - \frac{mc}{\hbar}\boldsymbol{\Psi} = 0, \qquad i\overline{\boldsymbol{\Psi}}_{\|\mu}\Upsilon^\mu + \frac{mc}{\hbar}\overline{\boldsymbol{\Psi}} = 0 \tag{2.13a, b}$$

The symmetric *energy-momentum tensor* $T^{\alpha\beta}$ is divergence free

$$T_{\alpha\beta} := -\frac{2}{\sqrt{-g}}\frac{\delta\mathscr{L}}{\delta g^{\alpha\beta}}, \qquad T^{\alpha\beta}{}_{\|\beta} = 0 \tag{2.14a, b}$$

$$T_{\alpha\beta} = t_{(\alpha\beta)}, \qquad t_{\alpha\beta} := \frac{\hbar}{2}(\overline{\boldsymbol{\Psi}}\Upsilon_\alpha\boldsymbol{\Psi}_{\|\beta} - \overline{\boldsymbol{\Psi}}_{\|\beta}\Upsilon_\alpha\boldsymbol{\Psi}) \tag{2.14c, d}$$

The *4-current* $j^\alpha$ is, because of (2.9) and (2.13), divergence free as well

$$j^\alpha := \overline{\boldsymbol{\Psi}}\Upsilon^\alpha\boldsymbol{\Psi}, \qquad j^\alpha{}_{\|\alpha} = 0 \tag{2.15a, b}$$

The Lagrange density $\mathscr{L}$ as well as $T^{\alpha\beta}, j^\alpha$ and the field equations (2.13) are, in consequence of (2.5) and (2.10c) with (2.6), invariant under the position dependent $\Omega_a{}^b$-**S**-transformations (2.2) and (2.10b). We mention the existence of an additional invariance under a global change of the representation of the $\Upsilon^a$-matrices:

$$\Upsilon^{a'} = \mathbf{V}\Upsilon^a\mathbf{V}^{-1}, \qquad \boldsymbol{\Psi}' = \mathbf{V}\boldsymbol{\Psi}, \qquad \overline{\boldsymbol{\Psi}}' = \overline{\boldsymbol{\Psi}}\mathbf{V}^{-1}, \qquad \frac{\partial\mathbf{V}}{\partial x^\alpha} = 0. \tag{2.16}$$

These *invariances of the theory*, together with the general coordinate invariant formulation, justify the choice of special coordinates, special $\Upsilon^a$ and a special tetrad field $h_a^\alpha(x)$ when actual calculations are carried out.

## 3. *Cosmological Reference Frame*

The evolution of the universe can be described by the kinematics of the cosmological substratum (approximating the motion of galactic clusters to that of a fluid). The 4-streamlines form a *congruence* of time-like worldlines

$$\chi^\alpha = \chi^\alpha(y^{\hat{a}}, ct), \qquad \hat{a} = 1, 2, 3 \tag{3.1}$$

$y^{\hat{a}}$ are three *scalars* labelling the respective worldline, $ct$ is its arc length. The normalised tangent vectors

$$u^\alpha := \frac{\partial x^\alpha}{\partial ct}, \qquad u^\alpha u_\alpha = 1 \tag{3.2}$$

describe the averaged 4-velocity of the cosmic matter.

‡ For the general case see Kofink (1949).

The isotropic homogeneous *Friedmann model* of assumption $A_4$ represents a good approximative description of the universe. It is characterised by‡

$$u_{\alpha\|\beta} = \tfrac{1}{3}\Theta(g_{\alpha\beta} - u_\alpha u_\beta), \qquad \Theta := u^\epsilon{}_{\|\epsilon} = \frac{3}{R}\frac{\partial R}{\partial ct}$$

The Hubble parameter $(1/R)(\partial R/\partial ct)$ represents the relative change in time of the distance between two galaxies (isotropic expansion). According to (3.3a) $u^\alpha$ is, because of $\omega^{\alpha\beta} = 0$, orthogonal to a family of space-like hypersurfaces $\Omega$. Furthermore, owing to $\dot{u}^\alpha = 0$, we can synchronise $t$ globally so that $\Omega$ denotes events of equal $t$ (cosmic time). The kinematics (3.3) of the cosmic masses determines the line-element of the space-time to be of the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = -R^2(t)\, d\sigma^2 + c^2\, dt^2 \tag{3.4}$$

where the $t$-independent $d\sigma$ is the line-element of a 3-space of constant curvature associated with $\Omega$. We restrict ourselves according to $A_4$, to the case of vanishing 3-curvature.

We can re-label the wordlines (3.1) by introducing parameters $y^{\hat{a}}$ with the properties: The $y^{\hat{a}}$-lines are geodesic and orthogonal to each other in the subspace $\Omega$ with regard of its metric $g_{\alpha\beta} - u_\alpha u_\beta$. Furthermore, we introduce a *tetrad field* $h^\alpha_{\hat{a}}(x)$ by adjusting its vectors as tangent vectors to the congruence (3.1) and the $y^{\hat{a}}$-lines in the subspaces:

$$h^\alpha_{(4)} = u^\alpha = \frac{\partial x^\alpha}{\partial ct}, \qquad h^\alpha_{\hat{a}} = \frac{1}{R}\frac{\partial x^\alpha}{\partial y^{\hat{a}}} \tag{3.5a, b}$$

With regard to subsequent *physical interpretations* we note that the $h^\alpha_{\hat{a}}$ represent the local frame of reference of observers which are, because of (3.5a), at rest relative to the galaxies, thus taking part at the cosmic expansion. The frames are globally synchronised by the cosmic time $t$ which is also the common proper time of the observers. The tetrads are Fermi propagated in time. The curves of constant observer numbers $y^{\hat{a}}$ with tangent vectors $h^\alpha_{\hat{a}}$ form, in the global rest space $\Omega$, a network of lines which are parallel and orthogonal to each other with regard to the metric in the rest space. The corresponding differential properties of the tetrad field as represented by the *Ricci rotation coefficients*

$$a_{abc} := h_{a\mu\|\nu} h^\mu_b h^\nu_c, \qquad a_{abc} = -a_{bac} \tag{3.6}$$

are

$$a_{(4)\hat{b}\hat{c}} = \frac{\Theta}{3}\eta_{\hat{b}\hat{c}}, \qquad \text{rest} = 0 \tag{3.7}$$

‡ (3.3) describes the following observational results: isotropic Hubble expansion ($\Theta := u^\epsilon{}_{\|\epsilon} = 3R^{-1}\partial R/\partial ct$), absence of non-gravitational forces between the galaxies ($\dot{u}^\alpha := u^\alpha{}_{\|\epsilon} u^\epsilon = 0$), shear-free galaxy motion ($\sigma_{\alpha\beta} := u_{(\alpha\|\beta)} - \dot{u}_{(\alpha} u_{\beta)} - \frac{1}{3}(g_{\alpha\beta} - u_\alpha u_\beta)\Theta = 0$), rotation-free galaxy motion ($\omega_{\alpha\beta} := u_{[\alpha\|\beta]} - \dot{u}_{[\alpha} u_{\beta]} = 0$).

We note for further reference that (3.7) implies ($\eta_{1234} = \sqrt{-g}$)‡

$$\eta^{\alpha\beta\kappa\lambda} h_{a\alpha\|\lambda} h_{(4)\beta} = 0 \tag{3.8}$$

Because of the invariance properties of the spinor theory we may specialise the $h_a^\alpha$-field of Section 2 to be the one introduced above. With

$$\mathbf{\Gamma}_\alpha = \tfrac{1}{4} a_{abc} h_\alpha^c \mathbf{\gamma}^b \mathbf{\gamma}^a \tag{3.9}$$

we obtain for $\mathbf{\Gamma}_{\hat{m}} := \mathbf{\Gamma}_\alpha h_{\hat{m}}^\alpha$

$$\mathbf{\Gamma}_{\hat{m}} = \frac{\Theta}{6} \mathbf{\gamma}_{\hat{m}} \mathbf{\gamma}^{(4)}, \qquad \mathbf{\Gamma}_{(4)} = 0 \tag{3.10}$$

and hence

$$\mathbf{\gamma}^{(4)} \mathbf{\Gamma}_{\hat{m}} + \mathbf{\Gamma}_{\hat{m}} \mathbf{\gamma}^{(4)} = 0, \qquad \mathbf{\gamma}^\alpha \mathbf{\Gamma}_\alpha = \frac{\Theta}{2} \mathbf{\gamma}^{(4)} \tag{3.11a, b}$$

## 4. *Hilbert Space*

We follow the outline given in Section 1.3 and construct the Hilbert space structure of our physical system out of the elements of the general relativistic theory of the Dirac field $\mathbf{\Psi}(x)$ described in Section 2. To begin with, we postulate a connection between the state vector $\langle \Psi \rangle$ with

$$\langle \Psi | \Psi \rangle = 1 \tag{4.1}$$

and the spinorial field $\mathbf{\Psi}(x)$.

We restrict the following to Dirac solutions which are square integrable in the sense that the integral ($d^3V$ is the invariant 3-volume element)

$$\int_\Omega j^\alpha h_{\alpha(4)}\, d^3V = \int_\Omega \mathbf{\Psi}^\dagger \mathbf{\Psi}\, d^3V \tag{4.2}$$

exists. *These* $\mathbf{\Psi}(x)$ *are, for fixed t, vectors of the function Hilbert space* $\mathsf{H}_f$ *with the interior product*

$$(\mathbf{\chi}, \mathbf{\theta}) := \int_\Omega \mathbf{\chi}^\dagger\, \mathbf{\theta}\, d^3V \tag{4.3}$$

$\mathbf{\Psi}$ vanishes because of (4.2) and $A_5$ at the surface $\sigma_\infty$ of $\Omega$ at infinity

$$\mathbf{\Psi}(\sigma_\infty) = 0 \tag{4.4}$$

‡ Furthermore we obtain from (3.7) that the $h_{\hat{a}}^\alpha$ are collinear to Killing vectors

(+) $\qquad \xi_{\hat{a}\alpha} := R h_{\hat{a}\alpha}, \qquad \xi_{\hat{a}(\alpha\|\beta)} = 0$

while $h_{(4)}^\alpha$ is collinear to a conformal Killing vector

(++) $\qquad \xi_{(4)\alpha} := R h_{(4)\alpha}, \qquad \xi_{(4)(\alpha\|\beta)} = (\partial R/\partial x^\epsilon h_{(4)}^\epsilon) g_{\alpha\beta}$

Furthermore $\mathbf{\Psi}$, as a Dirac solution, forms a divergence free current $j^{\alpha}$ (compare (2.15b)). This continuity equation leads, with Stokes's integral theorem, because of (4.4) and using (4.2) and (4.3) ($h^{\alpha}_{(4)}$ is the normal of $\Omega$), to

$$\frac{\partial}{\partial t}(\mathbf{\Psi},\mathbf{\Psi}) = 0 \tag{4.5}$$

*A normalisation*

$$(\mathbf{\Psi},\mathbf{\Psi}) = 1 \tag{4.6}$$

*of $\mathbf{\Psi}$ on one hypersurface $\Omega$ therefore remains preserved in time.*

The theory in Section 2 of Dirac solutions $\mathbf{\Psi}$ with (4.2) and (4.6) is the generalisation of the special relativistic Dirac theory of the Dirac particle in the Schrödinger picture and the Dirac spin-position representation. To incorporate flat space-time as a limiting case *we postulate that the A-component $\Psi_A$ ($A = 1, \ldots, 4$) of a normalised Dirac solution $\mathbf{\Psi}(x)$ at a point with observer parameters $y^{\hat{a}}$ on the hypersurface $t = const.$ is the projection of the time-dependent state vector $|\Psi\rangle = |\Psi\rangle_t$ (Schrödinger picture) on the Dirac spin-position eigenvector $|Y, A\rangle$ of a Hilbert space H:*

$$\Psi_A(y, t) = \langle \mathbf{A}, \mathbf{Y} | \Psi \rangle_t \tag{4.7}$$

*The state vector $|\Psi\rangle_t$ is accordingly attributed to the $\Omega$-hypersurface $t = const.$ as a whole.* H is the product space

$$H = H^{(O)} \times H^{(S)} \tag{4.8}$$

associated with the whole space-time and $|\mathbf{Y}, \mathrm{A}\rangle$ is obtained as the formal product

$$|\mathbf{Y}, \mathrm{A}\rangle = |\mathbf{Y}\rangle |\mathrm{A}\rangle \tag{4.9}$$

$\mathrm{H}^{(S)}$ is thereby the usual four-dimensional spin-variable space and $|\mathrm{A}\rangle$ its basis vectors (compare Messiah (1970)). Note the position independence (2.3b) of the $\gamma^a$. The $|\mathbf{Y}\rangle$ belong as unproper eigenvectors to the orbital-variable Hilbert space $\mathrm{H}^{(O)}$. They are eigenvectors of the Dirac position operator which has the scalar observer parameters $y^{\hat{a}}$ as the continuous eigenvalue spectrum.

Because of $\mathrm{H}_f$ being a representation of H, the $|\mathbf{Y}, \mathrm{A}\rangle$ form a complete basis of H

$$\sum_{A=1}^{4} \int |\mathbf{Y}, \mathrm{A}\rangle\langle \mathrm{A}, \mathbf{Y}| \, d\mu(y) = 1 \tag{4.10}$$

$\mu(y)$ is the measure function of the $y^{\hat{a}}$-spectrum. We obtain $\mu(y)$ from

$$(\mathbf{\Psi},\mathbf{\Psi}) = \langle \Psi | \Psi \rangle \tag{4.11}$$

with (4.3), (4.7) and inserting (4.10) ($d^3V = R^3(t)\, dy^{(1)}\, dy^{(2)}\, dy^{(3)}$)

$$d\mu = R^3(t)\, dy^{(1)}\, dy^{(2)}\, dy^{(3)} \tag{4.12}$$

*The measure function $\mu(y)$ of the $y^{\hat{a}}$-spectrum corresponding to the representation $\Psi(x)$ of $|\Psi\rangle$ is time-dependent.*

The general relativistic Dirac field $\Psi(x)$ is as a spinorial field element of the unconnected local tangent $u$ spaces of the manifold (compare Section 2). Accordingly $\Psi(x)$ shows the position dependent spin-transformation behaviour (2.2) with (2.4). This local dependence of $\Psi(x)$ does not make it necessary to attribute a different Hilbert space H or spin space $H^{(S)}$ to each point (as has been proposed by Utiyama (1965)) in order to be able to interpret $\Psi$ as a representation of a state vector. Instead *we can describe the $\Omega_a{}^b$-**S**-*transformations entirely in H *of (4.8) as*

$$|\Psi\rangle \rightarrow |\Psi\rangle' = U|\Psi\rangle \tag{4.13}$$

*with*

$$U = \sum_{A=1}^{4} \sum_{B=1}^{4} \int |Y, A\rangle\langle B, Y| S_{AB}(y, t)\, d\mu(y) \tag{4.14}$$

$U$ operates in $H^{(S)}$ as well as in $H^{(O)}$ and reduces only for $y^{\hat{a}}$-independent **S**-transformations to the special relativistic form. The unitarity $U^{\dagger} = U$ for tetrad rotations with fixed $h^{\alpha}_{(4)}$ follows for the infinitesimal case from (2.4) and (2.11).

## 5. *Evolution Operator*

Contraction of (4.10) with $\langle B, \mathbf{Y}|$ leads with (4.2) to the following orthonormality conditions for the $|\mathbf{Y}, B\rangle$

$$\langle B, Y'|Y, A\rangle = \delta_{AB}\delta^{(3)}(y - y')R^{-3}(t) \tag{5.1}$$

where $\delta^{(3)}$ is the usual three-dimensional Dirac function. It shows that the time-dependence of the measure function $\mu(y)$ implies a time-dependence of the basis vectors $|\mathbf{Y}, A\rangle = |\mathbf{Y}, A\rangle_t$. A representation of vectors and operators of H with regard to this basis contains therefore an additional time-dependence originating from the $|\mathbf{Y}, A\rangle$. To simplify the dynamics and especially to assimilate the description to the special relativistic one, we eliminate this additional time-dependence by *changing to a $y^{\hat{a}}$-representation with the trivial measure* $\mu(y) = d^3y := dy^{(1)}\, dy^{(2)}\, dy^{(3)}$. The corresponding basis vectors $|\mathbf{y}, A\rangle$ are complete and orthonormalised according to

$$\langle B, y'|y, A\rangle = \delta_{AB}\delta^{(3)}(y - y'), \qquad \sum_{A=1}^{4} \int |y, A\rangle\langle A, y|\, d^3y = 1 \tag{5.2a, b}$$

$|\mathbf{Y}, A\rangle$ and $|\mathbf{y}, A\rangle$ are associated with the same observer parameters $y^{\hat{a}}$ and therefore connected by

$$|\mathbf{y}, A\rangle = C|\mathbf{Y}, A\rangle \tag{5.3}$$

Comparison of (5.1) with (5.2) leads to

$$C = R(t)^{3/2} \tag{5.4}$$

where we have chosen the undetermined phase factor to be 1. (5.3) can be interpreted as a basis transformation. Its unitarity is guaranteed by the orthonormality relations (5.1) and (5.2a). *The two representations of the state vector* $|\Psi\rangle$ *are, because of (4.7) and (5.4), connected by*

$$\varphi_A(y, t) := \langle A, y\,|\Psi\rangle_t, \qquad \varphi_A(y, t) = R^{3/2}(t)\Psi_A(y, t) \tag{5.5a, b}$$

Accordingly, changing from $\boldsymbol{\Psi}$ to $\boldsymbol{\varphi}$ and using (3.11b) and (3.5), the Dirac equation (2.13a) becomes

$$\hbar_i \frac{\partial \boldsymbol{\varphi}}{\partial t} = \left(\frac{\hbar c}{i}\frac{1}{R}\boldsymbol{\alpha}^{\hat{a}}\frac{\partial}{\partial y^{\hat{a}}} + mc^2\boldsymbol{\beta}\right)\boldsymbol{\varphi} \tag{5.6}$$

with

$$\boldsymbol{\alpha}^{\hat{a}} := \boldsymbol{\gamma}^{(4)}\boldsymbol{\gamma}^{\hat{a}}, \qquad \boldsymbol{\beta} := \boldsymbol{\gamma}^{(4)} \tag{5.7}$$

The conservation (4.1) of probability in time leads for state vectors $|\Psi\rangle_t$ to a time evolution of the form

$$\frac{\partial\,|\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\mathscr{T}|\Psi\rangle, \qquad \mathscr{T}^\dagger = \mathscr{T} \tag{5.8a, b}$$

governed by a Hermitean evolution operator $\mathscr{T}$ (the dimension of $\mathscr{T}$ is that of an energy). Correspondingly the change in time of the expectation value $\bar{A} := {}_t\langle\Psi|\mathscr{A}|\Psi\rangle_t$ of an operator $\mathscr{A}$ is given by‡

$$\frac{\partial}{\partial t}\bar{A} = \frac{i}{\hbar}\langle\Psi|[\mathscr{T}, \mathscr{A}]\,|\Psi\rangle + \langle\Psi|\frac{\partial}{\partial t}\mathscr{A}|\Psi\rangle \tag{5.9}$$

We obtain $\mathscr{T}$ for the physical system in question *in postulating according to (1.2) that the general relativistic Dirac equation (5.6) represents the dynamical equation (5.9a) in the* $|y, A\rangle$*-representation.* Contraction of (5.9a) with the time-independent $\langle A, y\,|$ and comparison with (5.5a) and (5.6) gives the following representation of the *evolution operator*

$$\langle A, y\,|\mathscr{T}|\,y', B\rangle = \left(\frac{\hbar c}{i}\frac{1}{R(t)}\alpha^{\hat{m}}_{AB}\frac{\partial}{\partial y^{\hat{m}}} + mc^2\beta_{AB}\right)\delta^{(3)}(y - y') \tag{5.10}$$

The representation differs from the special relativistic one only by the presence of the factor $R(t)^{-1}$, neither the Hubble parameter $\Theta/3$ nor the derivatives of the metric occur.

‡ It is understood that we refer in the following statements about, $\mathscr{A}$, to $|\Psi\rangle_t$ from the domain of definition of $\mathscr{A}$. $[\mathscr{A}, \mathscr{B}] := \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}$.

## 6. *Energy*

We will introduce the operators,of energy, momentum and angular momentum in the Schrödinger picture according to (1.3) *in postulating that their mean values (expectation values) are the integrals of the corresponding classical expressions* $T_{\alpha\beta}a^{\alpha}h^{\beta}_{(4)}$ *(with* $T^{\alpha\beta}$ *of (2.14) and* $a^{\alpha}$ *appropriately chosen) over the hypersurface* $\Omega$. A symmetry property of the metrical field as expressed by a Killing vector $\xi^{\alpha}$ leads to a global conservation of the integral value of $T_{\alpha\beta}a^{\alpha}h^{\beta}_{(4)}$. The identification above ensures that, with regard to metric symmetries, the quantum mechanical mean values behave like the integral values of the respective classical quantities. A Killing vector field $\xi^{\alpha} = h^{\alpha}_{(4)}$ for instance would imply the conservation of the energy expectation value. In this sense the quantum mechanical mean values behave classically by construction, what may be interpreted as an *incorporation of a correspondence principle.*

To simplify these hypersurface-integrals we introduce

$$\hat{j}^{\alpha} := \bar{\Psi}\Upsilon^{\alpha}\Upsilon\Psi; \qquad \Upsilon := -\frac{i}{4!}\eta_{\alpha\beta\gamma\delta}\Upsilon^{\alpha}\Upsilon^{\beta}\Upsilon^{\gamma}\Upsilon^{\delta}, \qquad \Upsilon\Upsilon = 1 \quad (6.1\text{a–c})$$

and restrict to $\gamma^{a}$ which with (2.8a) obey not only (2.8b) but additionally

$$\Upsilon^{\alpha}\Upsilon^{\beta}\Upsilon^{\gamma} = \Upsilon^{\alpha}g^{\beta\gamma} + \Upsilon^{\gamma}g^{\alpha\beta} - \Upsilon^{\beta}g^{\gamma\alpha} + i\eta^{\alpha\beta\gamma\delta}\Upsilon_{\delta}\Upsilon \tag{6.2}$$

and therefore

$$\Upsilon^{\alpha}\Upsilon^{\beta} = g^{\alpha\beta} + \frac{i}{2}\eta^{\alpha\beta\gamma\delta}\Upsilon_{\gamma}\Upsilon_{\delta}\Upsilon \tag{6.3}$$

This special choice of the $\gamma^{a}$ is possible because of the invariance of the theory under $V$-transformations (2.16). With (6.2), (2.9) and because of $\eta^{\alpha\beta\gamma\delta}{}_{\|\epsilon} = 0$ we can write $t_{[\alpha\beta]}$ for Dirac solutions $\Psi$ as the divergence of a skew-symmetric tensor

$$t_{[\alpha\beta]} = -\frac{\hbar}{4}(\eta_{\alpha\beta}{}^{\kappa\lambda}\hat{j}_{\kappa})_{\|\lambda} \tag{6.4}$$

With the aid of Stokes's theorem we may therefore decompose the integral of

$$T_{\alpha\beta}a^{\alpha}h^{\beta}_{(4)} = (t_{\beta\alpha} + t_{[\alpha\beta]})a^{\alpha}h^{\beta}_{(4)} \tag{6.5}$$

over $\Omega$ with an arbitrary $a^{\alpha}$ as follows

$$\int_{\Omega} T_{\alpha\beta}a^{\alpha}h^{\beta}_{(4)}\, d^3V = J_1 + J_2 + J_3$$

$$J_1 := \int_{\Omega} t_{\alpha\beta}h^{\alpha}_{(4)}a^{\beta}\, d^3V, \qquad J_2 := \frac{\hbar}{4}\int_{\Omega} \hat{j}_{\kappa}(\eta^{\alpha\beta\kappa\lambda}a_{\alpha\|\lambda}h_{\beta(4)})\, d^3V$$

$$J_3 := \frac{\hbar}{4}\int_{\partial\Omega} \eta^{\alpha\beta\kappa\lambda}\hat{j}_{\kappa}a_{\alpha}n_{\lambda}h_{\beta(4)}\, d^2V = 0 \tag{6.6}$$

$O$ is thereby the 2-surface of the 3-hypersurface $\Omega$ at infinity. $n^\alpha$ is the unit vector normal to $O$ with $n_\alpha h^\alpha_{(4)} = 0$. $d^3V$ and $d^2V$ are the corresponding invariant infinitesimal 3- and 2-volumes. $J_3$ vanishes because of (4.4).

The mean value of the *energy* (kinetic plus rest-energy) as measured in the frame of the cosmic observers of Section 3 is obtained by choosing $a^\alpha = h^\alpha_{(4)}$. Because of (3.8) we have in this case $J_2 = 0$. Evaluation of the remaining integral $J_1$ with (2.14d) and (5.5b) gives

$$\int_\Omega T_{\alpha\beta}h^\alpha_{(4)}h^\beta_{(4)}\, d^3V = \frac{1}{2}\int \left[\boldsymbol{\varphi}^\dagger\left(-\frac{\hbar}{i}\frac{\partial}{\partial t}\boldsymbol{\varphi}\right) + \left(-\frac{\hbar}{i}\frac{\partial\boldsymbol{\varphi}}{\partial t}\right)^\dagger \boldsymbol{\varphi}\right] d^3y \tag{6.7}$$

We reformulate the right-hand side by means of elements of the Hilbert space. Because of (5.5a), (5.2) and (5.9a) together with the Hermitiecity (5.8b) of $\mathscr{T}$ we can write

$$\int_\Omega T_{\alpha\beta}h^\alpha_{(4)}h^\beta_{(4)}\, d^3V = \langle\Psi|\mathscr{T}|\Psi\rangle \tag{6.8}$$

*The operator of the relative energy is therefore equal to the evaluation operator* $\mathscr{T}$ *of (5.11).*

The development of the expectation value $\bar{E} = \langle\Psi|\mathscr{T}|\Psi\rangle$ in time is, according to (5.10), determined by $(\partial/\partial t)\mathscr{T}$. With the explicit representation (5.11) of $\mathscr{T}$ we obtain

$$\frac{\partial\bar{E}}{\partial t} = \left\langle\Psi\left|\frac{\partial}{\partial t}\mathscr{T}\right|\Psi\right\rangle = -\frac{1}{R}\frac{\partial R}{\partial t}\langle\Psi|\mathscr{T}|\Psi\rangle + \frac{1}{R}\frac{\partial R}{\partial t}mc^2\langle\Psi|\beta|\Psi\rangle \tag{6.9}$$

which reduces for vanishing rest mass $m = 0$ to the *$\bar{E}(t)$-law for neutrinos*‡

$$\frac{\partial\bar{E}}{\partial t} = -\frac{1}{R}\frac{\partial R}{\partial t}\bar{E}, \qquad \bar{E}(t) = \text{const.}\, R(t)^{-1} \tag{6.10}$$

This law is generally valid for arbitrary neutrino states $|\Psi\rangle$. *In the general case of massive particle $m \neq 0$ no $\bar{E}(t)$-law exists which is independent of the state vector* $|\Psi\rangle$. This will be discussed for the case of plane waves (eigenfunctions of the linear momentum) in a subsequent paper (Audretsch, 1973). We remark that the purely classical calculation, in which neutrinos are treated as structureless point-particles moving on null-geodesics, leads as well to the result (5.9).

## 7. *Linear Momentum*

We obtain the operator $\not{p}_{\hat{m}}$ of the $h^\alpha_{\hat{m}}$-component of the *linear momentum* by substituting $a^\alpha = -h^\alpha_{\hat{m}}$ in (6.6). $J_2$ vanishes because of (3.8). Using (3.5b),

‡ This result of a Hilbert space calculation is consistent with the purely field theoretica one: for $m = 0$ we have, with (2.13), $T^\alpha{}_\alpha = 0$. Because of (4.4) and (++) of footnote † (page 226) the value of $R\int_\Omega T_{\alpha\beta}h^\alpha_{(4)}h^\beta_{(4)}\, d^3V$ is independent of $\Omega$.

(3.11a) and (5.5b) we get for $J_1$

$$\bar{p}_{\hat{m}} = -\int_{\Omega} T_{\alpha\beta} h^{\alpha}_{\hat{m}} h^{\beta}_{(4)}\, d^3V = \frac{1}{2R} \int_{\Omega} \left[ \boldsymbol{\varphi}^{\dagger} \left( \frac{\hbar}{i} \frac{\partial \boldsymbol{\varphi}}{\partial y^{\hat{m}}} \right) + \left( \frac{\hbar}{i} \frac{\partial \boldsymbol{\varphi}}{\partial y^{\hat{m}}} \right)^{\dagger} \boldsymbol{\varphi} \right] d^3y \tag{7.1}$$

which leads to the following representation for the *momentum operator* $\not{p}_{\hat{m}}$

$$\langle A, y \,|\, \not{p}_{\hat{m}} \,|\, y', B \rangle = \frac{\hbar}{i} \frac{1}{R(t)} \frac{\partial}{\partial y^{\hat{m}}} \delta_{AB} \delta^{(3)}(y - y') \tag{7.2}$$

since $\not{p}_{\hat{m}}$ is Hermitean ($\not{p}_{\hat{m}}{}^{\dagger} = \not{p}_{\hat{m}}$).

The explicit representations (7.2) and (5.10) show that $\not{p}_{\hat{m}}$ commutes with $\mathscr{T}$

$$[\mathscr{T}, \not{p}_{\hat{m}}] = 0 \tag{7.3}$$

From (7.2) we obtain for $(\partial/\partial t)\, \not{p}_{\hat{m}}$

$$\left\langle A, y \left| \frac{\partial}{\partial t} \not{p}_{\hat{m}} \right| y, B \right\rangle = -\frac{1}{R} \frac{\partial R}{\partial t} \langle A, y \,|\, \not{p}_{\hat{m}} \,|\, y, B \rangle \tag{7.4}$$

We therefore obtain with (5.9) that *for arbitrary states* $|\Psi\rangle_t$ *the time-dependence of the* $h^{\alpha}_{\hat{m}}$*-components of the momentum is given by*‡

$$\frac{\partial \bar{p}_{\hat{m}}}{\partial t} = -\frac{1}{R} \frac{\partial R}{\partial t} \bar{p}_{\hat{m}}, \qquad \bar{p}_{\hat{m}}(t) = \text{const.}\, R(t)^{-1} \tag{7.5}$$

This result is in accordance with the one of the purely classical calculation for structureless point-particles moving on time-like geodesics.

Because of (7.2) the eigenfunction of $\not{p}_{\hat{m}}$ corresponding to the eigenvalue $p_{\hat{m}}$ is proportional to $e^{ip_{\hat{m}} y^{\hat{m}} R \hbar^{-1}}$ (no summation over $\hat{m}$). The wavelength $\lambda$ of this wave, as measured by the cosmic observers, is $R\Delta y^{\hat{m}}$ (compare (3.4)) with $p_{\hat{m}} \Delta y^{\hat{m}} R \hbar^{-1} = 2\pi$. For Dirac particles in an expanding universe the *De Broglie relation*

$$p = \frac{2\pi\hbar}{\lambda} \tag{7.6}$$

for eigenfunctions of the momentum therefore remains valid.

## 8. *Angular Momentum*

The special relativistic concept of the *total angular momentum* of a field in a space-like hypersurface $\Omega$ can easily be generalised to curved space-time in a covariant manner if the respective metric and topology are relatively simple. Consider in $\Omega$ a fixed base-point $P_0$ and an arbitrary point $P$, and let $e^{\mu}$ be a

‡ This result is again consistent with the corresponding field theoretical one, because (4.4) and (+) of footnote † (page 226) imply that the value of $R \int_{\Omega} T_{\alpha\beta} h^{\alpha}_{\hat{m}} h^{\beta}_{(4)}\, d^3V$ is independent of $\Omega$.

normalised vector at $P_0$ indicating a direction in $\Omega$. We join $P_0$ and $P$ by a 3-geodesic‡ $\mathscr{G}$ (its uniqueness is assumed), transport $e^\mu$ 3-parallel along $\mathscr{G}$ to $P$ and form at $P$ the tangent vector $r^\mu$ of $\mathscr{G}$ with $\sqrt{c - r^\mu r_\mu})$ being the geodesic 3-distance $\overline{P_0 P}$ along $\mathscr{G}$. By analogy with the special relativistic definition by a vector product, we now define at $P$ the density of the $e^\mu$-component of the angular momentum relative to the base-point $P_0$ by $J^\mu\, e_\mu$ with

$$J^\mu := -\eta^{\mu\gamma\alpha\beta} r_\gamma p_\alpha u_\beta \tag{8.1}$$

Hereby is $p^\alpha$ the density of the 4-momentum of the corresponding field

$$p^\alpha := T^{\alpha\beta} u_\beta, \qquad u^\alpha u_\alpha = 1 \tag{8.2}$$

and $u^\alpha$ the 4-velocity of the respective observer field which usually will be orthogonal to $\Omega$. *The $e^\mu$-component $J_e$ of the total angular momentum of the field with respect to an arbitrary base-point $P_0$ in the hypersurface $\Omega$ is obtained by integration over $\Omega$*

$$J_e = -\int_\Omega J^\mu\, e_u\, d^3V = -\int_\Omega T_{\alpha\beta} B^\alpha h^\beta_{(4)}\, d^3V \tag{8.3}$$

*with*

$$B^\alpha := -\eta^{\mu\gamma\alpha\beta}\, e_\mu r_\gamma u_\beta \tag{8.4}$$

Our cosmological space-time has Euclidian topology and the observer system in question is attributed hypersurface-orthogonally ($h^\alpha_{(4)} = u^\alpha$) to 3-flat hypersurfaces $\Omega$. Accordingly, taking the observer with $y^{\hat{a}} = 0$ as origin $P_0$ we get after an adjustment $x^{\hat{\alpha}} = y^{\hat{a}}, x^4 = ct$ of the coordinate system

$$r^\alpha = (y^{(1)}, y^{(2)}, y^{(3)}, 0), \qquad \sqrt{(-r^\alpha r_\alpha)} = \sqrt{(y^{(1)^2} + y^{(2)^2} + y^{(3)^2})} \tag{8.5}$$

To determine the $h^\alpha_{\hat{a}}$-components $J_{\hat{a}} := -J_\epsilon h^\epsilon_{\hat{a}}$ of $J^\mu$ we introduce $B^\mu_{\hat{a}}$ as $B^\mu$ constructed with $e^\mu = h^\mu_{\hat{a}}$. Evaluation of (8.4) leads to

$$\begin{aligned} B_{(1)\alpha} &= R^2(0, y^{(3)}, -y^{(2)}, 0) \\ B_{(2)\alpha} &= R^2(-y^{(3)}, 0, y^{(1)}, 0) \\ B_{(3)\alpha} &= R^2(y^{(2)}, -y^{(1)}, 0, 0) \end{aligned} \tag{8.6}$$

A straightforward calculation shows that the $B^\alpha_{\hat{a}}$ are Killing vectors

$$B_{\hat{a}(\alpha\|\beta)} = 0 \tag{8.7}$$

To calculate $J_{\hat{a}}$ according to (8.3) we make use of the decomposition (6.6). The structure (8.6) of $B^\alpha_{\hat{a}}$ involves

$$\eta^{\alpha\beta\gamma\delta} B_{\hat{a}\beta\|\gamma} h_{(4)\delta} = -2h^\alpha_{\hat{a}} \tag{8.8}$$

‡ The prefix '3-' denotes construction with regard to the metric of the three-dimensional subspace $\Omega$.

so that $J_2$ does not vanish. To evaluate $J_2$ we make use of $\bar{\mathbf{\Psi}} = \mathbf{\Psi}^\dagger h^\alpha_{(4)} \mathbf{\Upsilon}_\alpha$ (compare (2.10a) and (2.12)) and contract (6.3) with $\mathbf{\Upsilon}$. This leads with (6.1c) and (2.1a) to

$$\hat{j}_\kappa h^\kappa_{\hat{a}} = -\frac{2}{h} \mathbf{\Psi}^\dagger h^\kappa_{\hat{a}} \boldsymbol{\sigma}_\kappa \mathbf{\Psi} \tag{8.9}$$

where we have introduced

$$\boldsymbol{\sigma}^\alpha := \frac{\hbar}{4} \eta^{\alpha\beta\gamma\delta} \boldsymbol{\sigma}_{\beta\gamma} h_{(4)\delta}, \qquad \boldsymbol{\sigma}_{\mu\nu} := i \mathbf{\Upsilon}_{[\mu} \mathbf{\Upsilon}_{\nu]} \tag{8.10}$$

The space-like $\boldsymbol{\sigma}^\alpha$ has the characteristic properties (spin-$\frac{1}{2}$ particle)

$$(h^\kappa_{\hat{a}} \boldsymbol{\sigma}_\kappa)(h^\lambda_{\hat{a}} \boldsymbol{\sigma}_\lambda) = \left(\frac{\hbar}{2}\right)^2 \mathbf{1}, \qquad \sqrt{(-\boldsymbol{\sigma}^\alpha \boldsymbol{\sigma}_\alpha)} = \sqrt{(\tfrac{3}{4})}\hbar \tag{8.11}$$

For $\hat{a} = 1$ we have explicitly

$$h^\kappa_{(1)} \boldsymbol{\sigma}_\kappa = \boldsymbol{\sigma}_{(1)} = \frac{\hbar i}{2} \mathbf{\Upsilon}^{(2)} \mathbf{\Upsilon}^{(3)} \tag{8.12}$$

We complete the calculations of the example $\hat{a} = 1$. (2.14d) gives with (3.11a)

$$-t_{\beta\alpha} B^\alpha_{\hat{a}} h^\beta_{(4)} = \frac{\hbar}{i2} \left( \mathbf{\Psi}^\dagger \frac{\partial \mathbf{\Psi}}{\partial x^\beta} - \frac{\partial \mathbf{\Psi}^\dagger}{\partial x^\beta} \mathbf{\Psi} \right) B^\beta_{\hat{a}} \tag{8.13}$$

With (8.6), (5.5b) and

$$\frac{\hbar}{i} \left( y^{(2)} \frac{\partial}{\partial y^{(3)}} - y^{(3)} \frac{\partial}{\partial y^{(2)}} \right)$$

being the representation of an Hermitean operator we finally get

$$J_{(1)} = \int \boldsymbol{\varphi}^\dagger \left[ \frac{\hbar}{i} \left( y^{(2)} \frac{\partial}{\partial y^{(3)}} - y^{(3)} \frac{\partial}{\partial y^{(2)}} \right) + \frac{\hbar i}{2} \mathbf{\Upsilon}^{(2)} \mathbf{\Upsilon}^{(3)} \right] \boldsymbol{\varphi} \, d^3 y \tag{8.14}$$

*The operator $\mathscr{J}_{\hat{a}}$ of the $\hat{a}$-component of the total angular momentum is therefore equal to the special relativistic one*

$$\langle A, y | \mathscr{J}_{(1)} | y', B \rangle = \frac{\hbar}{i} \left( \delta_{AB} \left[ y^{(2)} \frac{\partial}{\partial y^{(3)}} - y^{(3)} \frac{\partial}{\partial y^{(2)}} \right] - \tfrac{1}{2} \sum_{C=1}^{4} \gamma^{(2)}_{AC} \gamma^{(3)}_{CB} \right) \times \delta^{(3)}(y - y') \tag{8.15}$$

with

$$[\mathscr{J}_{\hat{a}}, \mathscr{T}] = 0 \tag{8.16}$$

This implies, because of (5.10), the *conservation of the expectation value of the total angular momentum*

$$\frac{\partial}{\partial t} \bar{J}_{\hat{a}} = 0, \qquad \bar{J}_{\hat{a}} := \langle \Psi | \mathscr{J}_{\hat{a}} | \Psi \rangle \tag{8.17}$$

The time-dependence (7.5) of the linear momentum and the time-dependence of the 3-distance in the definition of the total angular momentum as a cross product (compare (8.1)) cancel each other.‡ The *operator of the spin-components* is, according to (8.12) and (8.15), equal to the special relativistic one ($\sigma^{\kappa}_{AB}$ are the components of $\sigma^{\kappa}$)

$$\langle A, y | \boldsymbol{\sigma}_{\hat{a}} | y', B \rangle = h_{\hat{a}\kappa} \sigma^{\kappa}_{AB} \delta^{(3)}(y - y') \tag{8.18}$$

## *Appendix*

*Alternative approaches.* One approach for constructing the elements of the above theory could consist of emphasising classical-quantum analogies (compare De Witt (1952)) by means of generalised coordinates and momenta. We do not follow the usual procedure of quantising a classical particle which leads from Poisson brackets to commutation relations, because no satisfactory general relativistic model of a classical particle with spin is availab.e We prefer to deduce the algebra of the observables which should reflect the gravitational interaction instead of taking it as a postulated starting point. Another possibility for constructing the dynamical equation is to start from the irreducible representations of the symmetry group of the respective curved space-time (Wigner approach). This has been discussed by many authors for the examples of the De Sitter and Einstein space (compare, e.g., Börner & Dürr (1969) and Kramer (1972)). We do not follow the lines of this approach because it enables no generalisation to less highly symmetric but physically more relevant space-times. Finally we have to mention that exact solutions of the general relativistic Dirac equation in a Robertson–Walker metric have been found for special matter distributions (eigenfunctions of the angular momentum) and special expansion laws ($R(t) = a + bt$ by Schrödinger (1940) and Taub (1937), $R(t) = e^{ct}$ by Taub (1937)). In this paper we have not been interested in obtaining further exact solutions but in discussing the structure of the theory and in deriving generally the behaviour of the fundamental observables.

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‡ (8.17) is, in the corresponding field theoretical calculation, a consequence of (8.3) and (8.7).

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